# LIMITS AND DERIVATIVES

**Limit :** If the right and left hand limits equal, then that common value is called the limit of $f(x)$ at $x = a$ and denote it by $\lim_{x\to a} f(x)$.

- $\lim_{x\to a^-} f(x)$ left hand limit of $f$ at $a$.
- $\lim_{x\to a^-} f(x)$ right hand limit of $f(x)$ at $a$.

**Theorem 1 :** Let $f$ and $g$ be two functions such that both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then

(i) $$\boxed{\lim_{x\to a}\left[f(x) \pm g(x)\right] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)}$$

(ii) $$\boxed{\lim_{x\to a}\left[f(x)\cdot g(x)\right] = \lim_{x\to a} f(x)\cdot \lim_{x\to a} g(x)}$$

(iii) $$\boxed{\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a}}{\lim_{x\to a}}}$$

**Theorem 2 :** For any positive integer $n$, $$\boxed{\lim_{x\to a} \frac{x^n - a^n}{x-a} = na^{n-1}}$$

**Theorem 3 :** Let $f$ and $g$ be any two real valued functions with the same domain such that $f(x) \le g(x)$ for all $x$ in the domain of definition, For some $a$, if both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then $\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$.

**Theorem 4 :** (Sandwich theorem) : Let $f$, $g$ and $h$ be real functions such that $f(x) \le g(x) \le h(x)$ for all $x$ in the common domain of definition. For some real number $a$, if $\lim_{x\to a} f(x) = l = \lim_{x\to a} h(x)$, then $\lim_{x\to a} g(x) = l$.

**Theorem 5 :** The following are two important limits (i) $\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1}$ (ii) $\boxed{\lim_{x\to 0} \frac{1-\cos x}{x} = 0}$

**Derivative :** The derivative of a function $f$ at $a$ is defined by $$\boxed{f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}}$$

Derivative of a function $f$ at a point $x$ is defined by $$\boxed{f'(x) = \frac{df(x)}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}}$$

first principle of derivative

**Note :** $$\boxed{\lim_{x\to a}\left[(\lambda\cdot f)(x)\right] = \lambda \lim_{x\to a} f(x)}$$

**Limits of polynomials and rational functions :** A function $f$ is said to be a polynomial function if $f(x)$ is zero function or if $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots\ldots + a_n x^n$. where $a_i$s are real numbers such that $a_n \ne 0$ for some natural number $n$.

**Theorem 6 :** For functions $u$ and $v$ the following holds : (Leibnitz rule)

(i) $(u \pm v)' = u' \pm v'$ (ii) $(uv)' = u'v + uv'$ (iii) $\left[\frac{u}{v}\right]' = \frac{u'v - uv'}{v^2}$ provided all are defined

**Theorem 7 :** $f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$ for any positive integer $n$.

**Theorem 8 :** $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ be a polynomial function where $a_i$s are all real numbers and $a_n \ne 0$. Then the derivative function is given by

$$\boxed{\frac{df(x)}{dx} = na_n x^{n-1} + (n-1)a_{n-1} x^{x-2} + \ldots\ldots + 2a_2 x + a_1}$$

- $\boxed{\frac{d}{dx}(x^n) = nx^{n-1}}$
- $\boxed{\frac{d}{dx}(\sin x) = \cos x}$
- $\boxed{\frac{d}{dx} = -\sin x}$